Compressible Jeffery-Hamel Flow of a Plasma

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A similarity transformation is given, which reduces the partial, nonlinear differential equations describing a compressible, polytropic plasma flow across an azimuthal magnetic field in a duct with plane inclined walls to an ordinary nonlinear differential equation of second order. The latter is solved rigorously in terms of a hyperelliptic integral. The form of the plasma flow fields in pure outflows (diffuser) is discussed analytically in dependence of the Reynolds (R) and Hartmann (H) numbers and the polytropic coefficient (γ) for given duct angles θ_0. The realizable Mach numbers are shown to be eigenvalues of the nonlinear boundary-value problem, $M = M_\varkappa(R, H, \gamma, \theta_0)$. The flow solutions are different in type for Hartmann numbers H 1) below and 2) above a critical Hartmann number H_c defined by $H_c^2 = [2(\gamma-1)/(\gamma+1)]R + [2\gamma/(\gamma+1)]^2$. Some of the eigenvalues $M_\varkappa$ are calculated and the associated velocity profiles are represented graphically for prescribed flow parameters.

I. Introduction

The resolution of nonlinear partial differential equations can frequently be reduced to the analysis of ordinary nonlinear differential equations by means of so-called similarity transformations [1]. Thus, rigorous solutions have been found in the theory of incompressible, viscous fluids [2, 3] and compressible, nonviscous gases [4−7]. This investigation is concerned with the divergent, compressible flow of a viscous plasma across an azimuthal magnetic field in a duct with plane, inclined walls (Jeffery-Hamel flow) [2, 3]. As usual in the mathematical theory of compressible gases and plasma [8, 9], conservation of energy is taken into account by means of a polytropic energy integral with polytropic coefficient $\gamma\,(0 \leqq \gamma \leqq \infty$, $\gamma = 0$, 1, and ∞ correspond to isobaric, isothermal and incompressible flow, respectively).

It is shown that a similarity transformation of the structure $F(r,\theta) = r^{-m}\,G(\theta)$ exists for the plasma fields (with different power "m" and function "G" for different fields) which transforms the nonlinear partial differential equations of the compressible flow into an ordinary nonlinear differential equation of second order for the amplitude $g(\theta)$ of the radial velocity field, $u(r,\theta) = r^{-n}\,g(\theta)$. The solution of the latter differential equation is given in terms of a hyperelliptic integral which is evaluated analytically for large Reynolds numbers as of physical interest. The theory presents information on the dependence of the velocity, pressure and density distributions on the Reynolds (R), Hartmann (H), polytropic (γ) numbers and duct angles (θ_0) in outflows. The Mach number is shown to be an eigenvalue of the nonlinear boundary-value problem,

$$M = M_\varkappa(R, H, \gamma, \theta_0)\,.$$

In the limiting case of vanishing electrical conductivity or magnetic field, the similarity theory describes the compressible, viscous flow of an ordinary gas in a diffuser, a previously unsolved problem of gas dynamics. If, in addition, the limiting process, $\gamma \to \infty$, is carried through, the solution reduces to one obtained in connection with the classical investigations on the incompressible outflow of a viscous fluid between inclined walls by Jeffery and Hamel [2, 3].

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II. Formulation of Problem

Let cylindrical coordinates (r, θ, z) be introduced for the description of the Jeffery-Hamel plasma flow (Figure 1). The plasma flow is bounded in the planes $(\theta = +\theta_0,\ r_1 \leqq r \leqq r_2)$ and $(\theta = -\theta_0,\ r_1 \leqq r \leqq r_2)$ by insulating walls, and quasi-unbounded in the directions parallel to the z-axis. The latter assumption is applicable to a finite diffuser with electrode plates at $z = \pm z_\infty$, where $z_\infty \gg {}^1\!/_2(r_1 + r_2)\,\theta_0$. The injection $(r = r_1)$ and removal $(r = r_2)$ of the plasma occurs in a selfsimilar way. In fluid dynamic experiments, this is commonly realized by putting the diffuser into a much larger, similar diffuser through which the working gas is pumped at the desired rate [10] (Figure 1).

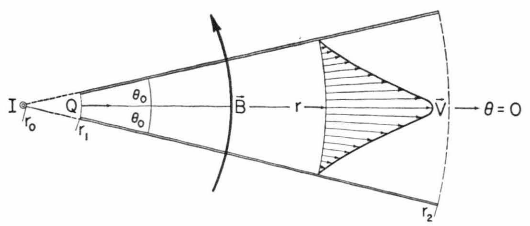

Fig. 1. Geometry of duct with azimuthal magnetic field (qualitative velocity profile).

The magnetic field has its sources in an electric current I flowing through a conducting rod ($0 \leqq r \leqq r_0$, $-\infty \leqq z \leqq +\infty$, $r_0 < r_1$). In accordance with Stokes' law, $\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = \mu_0 I$, the magnetic field is azimuthal (μ_0 = permeability of vacuum), and has the induction

$$\boldsymbol{B} = \frac{\mu_0}{2\pi} \frac{I}{r} \boldsymbol{e}_\theta, \quad r_0 \leqq r < \infty. \tag{1}$$

In absence of flow sources or sinks at the inclined walls, the velocity field is radial, if the accelerating force fields are radial,

$$\boldsymbol{v} = u\, \boldsymbol{e}_r. \tag{2}$$

The flow of the plasma (conductivity σ) across the magnetic field induces an axial current density field (Hall-effect disregarded) [11]

$$\boldsymbol{j} = \sigma (E_z + u B_\theta) \boldsymbol{e}_z. \tag{3}$$

The resulting Lorentz force density is a purely radial field which opposes the inducting flow,

$$\boldsymbol{j} \times \boldsymbol{B} = -\sigma (E_z + u B_\theta) B_\theta \boldsymbol{e}_r. \tag{4}$$

Because of $\nabla \times \boldsymbol{E} = \boldsymbol{0}$, $\nabla \cdot \boldsymbol{j} = \sigma(\nabla \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \nabla \times \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \times \boldsymbol{B}) = \sigma \nabla \cdot \boldsymbol{E} = 0$, and the boundary conditions at $z = z_{\pm\infty}$, the electric field vanishes,

$$\boldsymbol{E} = E_z \boldsymbol{e}_z = \boldsymbol{0}, \quad \text{for} \quad \boldsymbol{E}_{z=z_{\mp\infty}} = \boldsymbol{0}. \tag{5}$$

The axial current density j_z flows through the planes $z = \pm z_\infty$ (electrodes) and forms an electrical current J in the external circuit,

$$J = I \frac{\mu_0 \sigma}{2\pi} \int_{r_1}^{r_2} \int_{-\theta_0}^{+\theta_0} u(r, \theta)\, \mathrm{d}r\, \mathrm{d}\theta. \tag{6}$$

The Eqs. (1) – (6) are based on the assumption that the induced magnetic field is small compared to the external magnetic field, which implies small magnetic Reynolds numbers,

$$R_B = \mu_0 \sigma u(r, \theta) r \ll 1, \quad r_1 \leqq r \leqq r_2, \quad |\theta| \leqq \theta_0.$$

This condition is satisfied in many experiments, operating at temperature levels, for which the plasma is weakly or partially ionized.

After the preparations in Eqs. (1) – (6), the nonlinear boundary-value problem describing the velocity [$u = u(r, \theta)$], density [$\varrho = \varrho(r, \theta)$], and pressure [$p = p(r, \theta)$] fields of the steady-state plasma flow between inclined ($-\theta_0 \leqq \theta \leqq \theta_0$) walls can be formulated as:

$$\varrho u \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} \right] + \frac{\mu}{3} (1 + \varepsilon) \frac{\partial}{\partial r} \left[\frac{\partial u}{\partial r} + \frac{u}{r} \right] - \sigma \left(\frac{\mu_0 I}{2\pi} \right)^2 \frac{u}{r^2}, \tag{7}$$

$$0 = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{2\mu}{r^2} \frac{\partial u}{\partial \theta} + \frac{\mu}{3} (1 + \varepsilon) \frac{\partial}{r\, \partial \theta} \left[\frac{\partial u}{\partial r} + \frac{u}{r} \right], \tag{8}$$

$$\partial (\varrho u) / \partial r + \varrho u / r = 0, \tag{9}$$

$$p = \alpha \varrho^\gamma, \quad \alpha \equiv p_0 / \varrho_0^\gamma, \quad (u (\partial p / \partial r) = -\gamma p [\partial u / \partial r + u / r]), \tag{10}$$

where

$$u(r, \theta)_{\theta = \pm\theta_0} = 0, \quad u(r, +\theta) = u(r, -\theta) \geqq 0, \quad r_1 \leqq r \leqq r_2, \tag{11}$$

$$u(r, \theta)_{r = r_0,\, \theta = \bar{\theta}} = u_0 > 0, \quad r_1 \leqq r_0 \leqq r_2, \quad -\theta_0 < \bar{\theta} < +\theta_0, \tag{12}$$

and

$$Q = \int_{-\theta_0}^{+\theta_0} (\varrho u) r\, \mathrm{d}\theta, \quad Q > 0; \quad u(r, \theta)_{r = r_{1,2}} = g(\theta) r_{1,2}^{-m}, \quad -\theta_0 \leqq \theta \leqq +\theta_0. \tag{13}$$

In the solution of this boundary-value problem, two integration constants [$g(\tilde{\theta}), C$] and one separation constant ($\varkappa$) appear, which are determined by Equations (11) – (12). Equation (11) contains the boundary conditions for zero slip at the walls and the symmetry conditions. Equation (12) specifies that the velocity at an appropriate point ($r_0, \bar{\theta}$) of the flow is prescribed, $u(r_0, \bar{\theta}) = u_0 > 0$.

In Equation (13), the first relation gives the flow rate Q per unit length ($\Delta z = 1$) associated with u_0, and vice versa [$Q = Q(u_0)$], and the second relation specifies that the plasma has to be injected at some nonvanishing radius $r = r_1 > 0$ and extracted at some finite radius $r = r_2 < \infty$ in a selfsimilar way. $g(\theta)$ represents the azimuthal dependence of similarity solution to be derived.

In the momentum conservation equation [Eqs. (7) – (8)], (nondiagonal) shear stresses ($\mu > 0$)

and (diagonal) secondary viscosity stresses ($\eta = \varepsilon\,\mu/3$, $\varepsilon \ll 1$ for rarefied mono-atomic systems) are considered. The compressible plasma flow exhibits a convective nonlinearity [Eq. (7)], and additional nonlinearities through the conservation equations for mass [Eq. (9)] and energy [Eq. (10)] in the polytropic approximation ($a = p_0/\varrho_0{}^\gamma$ is invariant). The polytropic number γ is a phenomenological plasma parameter, except for quasi-adiabatic conditions where $\gamma \cong c_p/c_v$ [9].

For mathematical convenience, the nonlinear boundary-value problem in Eqs. (7) – (13) is reformulated in terms of dimensionless variables ($\sim$) as:

$$\tilde{\varrho}\,\tilde{u}\,\frac{\partial \tilde{u}}{\partial \tilde{r}} = -\frac{1}{\gamma M^2}\frac{\partial \tilde{p}}{\partial \tilde{r}} + \frac{1}{R}\left[\frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{1}{\tilde{r}^2}\frac{\partial^2 \tilde{u}}{\partial \theta^2} - \frac{\tilde{u}}{\tilde{r}^2}\right] + \frac{1}{3}\left(\frac{1+\varepsilon}{R}\right)\frac{\partial}{\partial \tilde{r}}\left[\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{\tilde{u}}{\tilde{r}}\right] - \frac{H^2}{R}\frac{\tilde{u}}{\tilde{r}^2}, \quad (14)$$

$$0 = -\frac{1}{\gamma M^2}\frac{1}{\tilde{r}}\frac{\partial \tilde{p}}{\partial \theta} + \frac{2}{R}\frac{1}{\tilde{r}^2}\frac{\partial \tilde{u}}{\partial \theta} + \frac{1}{3}\left(\frac{1+\varepsilon}{R}\right)\frac{\partial}{\tilde{r}\,\partial \theta}\left[\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{\tilde{u}}{\tilde{r}}\right], \quad (15)$$

$$\partial(\tilde{\varrho}\,\tilde{u})/\partial\tilde{r} + \tilde{\varrho}\,\tilde{u}/\tilde{r} = 0, \quad (16)$$

$$\tilde{p} = \tilde{\varrho}^\gamma, \quad (\tilde{u}(\partial\tilde{p}/\partial\tilde{r}) = -\gamma\,\tilde{p}[\partial\tilde{u}/\partial\tilde{r} + \tilde{u}/\tilde{r}]), \quad (17)$$

where

$$\tilde{u}(\tilde{r},\theta)_{\theta=\pm\theta_0} = 0, \quad \tilde{u}(\tilde{r},+\theta) = \tilde{u}(\tilde{r},-\theta) \geqq 0, \quad \tilde{r}_1 \leqq \tilde{r} \leqq \tilde{r}_2, \quad (18)$$

$$\tilde{u}(\tilde{r},\theta)_{\tilde{r}=1,\theta=\bar{\theta}} = 1, \quad -\theta_0 < \bar{\theta} < +\theta_0, \quad (19)$$

and

$$\tilde{Q} = \int_{-\theta_0}^{+\theta_0}(\tilde{\varrho}\,\tilde{u})\,\tilde{r}\,\mathrm{d}\theta, \quad \tilde{Q} = Q/\varrho_0\,u_0\,r_0; \quad \tilde{u}(\tilde{r},\theta)_{\tilde{r}=\tilde{r}_{1,2}} = \tilde{g}(\theta)\,\tilde{r}_{1,2}^{-m}, \quad -\theta_0 \leqq \theta \leqq +\theta_0. \quad (20)$$

In Eqs. (14) – (20), the Hartmann (H), Mach (M), and Reynolds (R) numbers, and the dimensionless variables $\tilde{p}$, $\tilde{\varrho}$, $\tilde{u}$, $\tilde{g}$ and $\tilde{r}$ are defined by

$$H^2 = (\sigma/\mu)\,(B_\theta\,r)^2 = (\sigma/\mu)\,(\mu_0\,I/2\,\pi)^2, \quad (21)$$

$$M^2 = u_0{}^2/(\gamma\,p_0/\varrho_0); \qquad R = u_0\,r_0/(\mu/\varrho_0), \quad (22); \quad (23)$$

and

$$\tilde{p} = p/p_0, \quad \tilde{\varrho} = \varrho/\varrho_0, \quad \tilde{u} = u/u_0, \quad \tilde{g} = g/g_0, \quad \tilde{r} = r/r_0; \quad p_0 = p(r_0,\bar{\theta}) \neq 0, \quad (24)$$

$$\varrho_0 = \varrho(r_0,\bar{\theta}) \neq 0, \; u_0 = u(r_0,\bar{\theta}) \neq 0, \; g_0 = g(\bar{\theta}) \neq 0.$$

It should be noted that u_0 and p_0 are not independent for given values of M and R, and vice versa, since $M^2/R = (\mu/\gamma\,r_0)\,u_0/p_0$. A suitable choice of reference values for applications is, e. g.,

$$p_0 = p(r_0,0), \quad \varrho_0 = \varrho(r_0,0), \quad u_0 = u(r_0,0), \; g_0 = g(0), \quad (25)$$

since the flow values at a fixed point r_0 along the central streamline $\theta = 0$ are most likely those to be known from experiments.

The designations of dimensionless variables by a tilde is dispensed with in the following.

III. Similarity Solution

The continuity Eq. (16) indicates that the flow density field $\varrho\,u$ is a selfsimilar function of r and θ,

$$\varrho(r,\theta)\,u(r,\theta) = r^{-1}f(\theta), \quad (26)$$

where $f(\theta)$ has been generated by integration with respect to r. For the velocity field u, a similarity ansatz is made in the form

$$u(r,\theta) = r^{-m}\,g(\theta)\,. \quad (27)$$

It is seen by comparison of Eqs. (26) and (27), that the density field has the selfsimilar structure

$$\varrho(r,\theta) = r^{m-1}f(\theta)/g(\theta)\,. \quad (28)$$

Hence,

$$p(r,\theta) = r^{\gamma(m-1)}[f(\theta)/g(\theta)]^\gamma \quad (29)$$

by Eq. (17) for the selfsimilar pressure field. Substitution of Eqs. (27) – (28) into Eq. (15) yields

$$r^{\gamma(m-1)-1}(f/g)^{\gamma-1}\mathrm{d}(f/g)/\mathrm{d}\theta = \tfrac{1}{3}(M^2/R) \cdot [6-(m-1)(1+\varepsilon)]\,r^{-(m+2)}\,\mathrm{d}g/\mathrm{d}\theta\,. \quad (30)$$

A selfsimilar solution exists by Eq. (30) if the power m is specified as

$$m = (\gamma-1)/(\gamma+1), \quad -1 \leqq m \leqq +1 \quad \text{for} \quad 0 \leqq \gamma \leqq \infty. \quad (31)$$

According to Eqs. (30) and (31), the differential relation between $f(\theta)$ and $g(\theta)$ is

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{f}{g}\right)^\gamma = \frac{2}{3}\,\gamma\,\frac{M^2}{R}\left[3 + \frac{1+\varepsilon}{1+\gamma}\right]\frac{\mathrm{d}g}{\mathrm{d}\theta}\,. \quad (32)$$

Hence,

$$\frac{f}{g} = \left[a + \frac{2}{3}\,\gamma\,\frac{M^2}{R}\left(3 + \frac{1+\varepsilon}{1+\gamma}\right)g\right]^{1/\gamma}, \quad (33)$$

where $a = a(\gamma)$ is an integration constant. Insertion of the Eqs. (31) and (33) into the Eqs. (27) – (29)

leads to the intermediate result:

$$u(r,\theta) = r^{-(\gamma-1)/(\gamma+1)}\, g(\theta), \tag{34}$$

$$\varrho(r,\theta) = r^{(\gamma-1)/(\gamma+1)-1}\left[a + \frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\right]^{1/\gamma}, \tag{35}$$

$$p(r,\theta) = r^{\gamma((\gamma-1)/(\gamma+1)-1)}\left[a + \frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\right]. \tag{36}$$

Integration of Eq. (15) with respect to θ, and elimination of the pressure field from Eq. (14) yields the equations

$$p = 2\gamma\frac{M^2}{R}\frac{u}{r} + \frac{1}{3}\gamma\frac{M^2}{R}(1+\varepsilon)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right) + \beta(r), \tag{37}$$

and

$$\varrho\, u\frac{\partial u}{\partial r} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{u}{r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right) - \frac{H^2}{R}\frac{u}{r^2} - \frac{1}{\gamma M^2}\frac{\mathrm{d}\beta(r)}{\mathrm{d}r}. \tag{38}$$

Hence,

$$p = \frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\, r^{-(2\gamma)/(\gamma+1)} + \beta(r), \tag{39}$$

and

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} + \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]g + \frac{\gamma-1}{\gamma+1}R\cdot \cdot\left[a + \frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\right]^{1/\gamma} g^2 = = \frac{R}{\gamma M^2} r^{(3\gamma+1)/(\gamma+1)}\frac{\mathrm{d}\beta(r)}{\mathrm{d}r} \equiv -\varkappa R, \tag{40}$$

by substitution for $u(r,\theta)$ and $p(r,\theta)$ in accordance with Eqs. 34), (36). The product $-\varkappa R$ designates the separation invariant of Equation (40). The r-dependent pressure constituent obtains by integration as

$$\beta(r) = \tfrac{1}{2}(\gamma+1)\varkappa M^2 r^{-2\gamma/(\gamma+1)}. \tag{41}$$

From the identity of Eqs. (36) and (39) it follows that the similarity transformation does not permit a homogeneous over-pressure [12], and that

$$a = \tfrac{1}{2}(\gamma+1)M^2\varkappa, \quad \varkappa \sim R^0, \quad \varkappa > 0\,(p>0). \tag{42}$$

Equation (42) indicates that the integration constant $a = a(\gamma)$ [Eq. (33)] and the separation constant $\varkappa$ [Eq. (40)] are proportional.

According to the Eqs. (40) and (42), and the conditions in Eqs. (18) – (20), the function $g(\theta)$ is described by the nonlinear boundary-value problem:

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} + \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]g + \frac{\gamma-1}{\gamma+1}R\left[a + \frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\right]^{1/\gamma} g^2 + 2\frac{M^{-2}R}{\gamma+1}a = 0 \tag{43}$$

where

$$g(\theta)_{\theta=\pm\theta_0} = 0, \quad g(+\theta) = g(-\theta); \quad g(\bar\theta) = 1, \quad -\theta_0 < \bar\theta < +\theta_0, \qquad (44);\ (45)$$

and

$$Q = \int_{-\theta_0}^{+\theta_0}\left[a + \frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\right]^{1/\gamma} g\,\mathrm{d}\theta \tag{46}$$

are the conditions determining the integration $[C, g(\theta)]$ and separation $(a \sim \varkappa)$ constants, and the flow rate associated with the selfsimilar flow, respectively.

Equation (43) is readily integrated by separation of variables, which leads to the differential expression

$$\frac{1}{2}d\left(\frac{\mathrm{d}g}{\mathrm{d}\theta}\right)^2 = -\left\{\left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]g + \frac{\gamma-1}{\gamma+1}R\left[a + \frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\right]^{1/\gamma} g^2 + 2\frac{M^{-2}R}{\gamma+1}a\right\}\mathrm{d}g.$$

Hence,

$$\begin{aligned}\left(\frac{\mathrm{d}g}{\mathrm{d}\theta}\right)^2 = {} & C - 4\frac{M^{-2}R}{\gamma+1}\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right]a^* g - \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]g^2 \\ & - 2\frac{\gamma-1}{\gamma+1}R\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\gamma}{1+\gamma}\right)\right]^{1/\gamma}\left\{\frac{\gamma}{1+3\gamma}\left[(a^*+g)^{(1+3\gamma)/\gamma} - a^{*(1+3\gamma)/\gamma}\right]\right. \\ & \left. - 2a^*\frac{\gamma}{1+2\gamma}\left[(a^*+g)^{(1+2\gamma)/\gamma} - a^{*(1+2\gamma)/\gamma}\right] + a^{*2}\frac{\gamma}{1+\gamma}\left[(a^*+g)^{(1+\gamma)/\gamma} - a^{*(1+\gamma)/\gamma}\right]\right\}\end{aligned} \tag{47}$$

by ordinary integrations, where C is an integration constant, and a^* a parameter which is large in all cases of interest,

$$a^* \equiv a\Big/\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right] \propto R \gg 1 \tag{48}$$

by Equation (42). Another integration gives an implicit, closed form solution for $g(\theta)$ in terms of the quadrature:

$$\int_{g(\tilde{\theta})}^{g(\theta)} \mathrm{d}g/\sqrt{P(g)} = \pm \int_{\tilde{\theta}}^{\theta} \mathrm{d}\theta\,, \tag{49}$$

where

$$\begin{aligned} P(g) = C &- 4\frac{M^{-2}R}{\gamma+1}\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right]a^* g - \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]g^2 \\ &- 2\frac{\gamma-1}{\gamma+1}R\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right]^{1/\gamma}\left\{\frac{\gamma}{1+3\gamma}\left[(a^*+g)^{(1+3\gamma)/\gamma} - a^{*(1+3\gamma)/\gamma}\right]\right. \\ &\left. - 2a^*\frac{\gamma}{1+2\gamma}\left[(a^*+g)^{(1+2\gamma)/\gamma} - a^{*(1+2\gamma)/\gamma}\right] + a^{*2}\frac{\gamma}{1+\gamma}\left[(a^*+g)^{(1+\gamma)/\gamma} - a^{*(1+\gamma)/\gamma}\right]\right\} \end{aligned} \tag{50}$$

and "$\pm$" sign for $\mathrm{d}g/\mathrm{d}\theta \gtrless 0$.

In Eq. (49), $\tilde{\theta}$ is an arbitrary reference angle, $-\theta_0 \leqq \tilde{\theta} \leqq \theta_0$, and the associated $g(\tilde{\theta})$ the remaining integration constant. Similarly, the integration constant C is related to the root $\hat{g}$, for which $P(\hat{g}) = [\mathrm{d}g/\mathrm{d}\theta]^2_{g=\hat{g}} = 0$, by

$$\begin{aligned} C = 4&\frac{M^{-2}R}{\gamma+1}\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right]a^*\hat{g} + \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]\hat{g}^2 \\ &+ 2\frac{\gamma-1}{\gamma+1}R\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right]^{1/\gamma}\left\{\frac{\gamma}{1+3\gamma}\left[(a^*+\hat{g})^{(1+3\gamma)/\gamma} - a^{*(1+3\gamma)/\gamma}\right]\right. \\ &\left. - 2a^*\frac{\gamma}{1+2\gamma}\left[(a^*+\hat{g})^{(1+2\gamma)/\gamma} - a^{*(1+2\gamma)/\gamma}\right] + a^{*2}\frac{\gamma}{1+\gamma}\left[(a^*+\hat{g})^{(1+\gamma)/\gamma} - a^{*(1+\gamma)/\gamma}\right]\right\}. \end{aligned} \tag{51}$$

The general solution given in Eq. (49) represents a hyperelliptic integral [13] for any $\gamma \neq 0, 1, \infty$.

A binomial expansion of the transcendental term in Eq. (43) leads to the semi-convergent expansions for $P(g)$ and C:

$$\begin{aligned} P(g) = C &- 4\frac{M^{-2}R}{\gamma+1}\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right]a^* g - \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]g^2 \\ &- \frac{2}{3}\frac{\gamma-1}{\gamma+1}R\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)a^*\right]^{1/\gamma}\left\{1+\sum_{n=1}^{\infty}\frac{3}{3+n}\binom{\gamma^{-1}}{n}\left(\frac{g}{a^*}\right)^n\right\}g^3, \quad |g/a^*|<1, \end{aligned} \tag{52}$$

and

$$\begin{aligned} C = 4&\frac{M^{-2}R}{\gamma+1}\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\right]a^*\hat{g} + \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]\hat{g}^2 \\ &+ \frac{2}{3}\frac{\gamma-1}{\gamma+1}R\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)a^*\right]^{1/\gamma}\left\{1+\sum_{n=1}^{\infty}\frac{3}{3+n}\binom{\gamma^{-1}}{n}\left(\frac{\hat{g}}{a^*}\right)^n\right\}\hat{g}^3, \quad |\hat{g}/a^*|<1. \end{aligned} \tag{53}$$

By combining Eqs. (52), (53) and (48), one obtains for $P(g)/\hat{g}^2$ the expression ($\hat{g} \neq 0$)

$$\begin{aligned} \frac{P(g)}{\hat{g}^2} &= 4\frac{M^{-2}R}{\gamma+1}\left(\frac{a}{\hat{g}}\right)\left[1-\frac{g}{\hat{g}}\right] + \left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]\left[1-\left(\frac{g}{\hat{g}}\right)^2\right] \\ &+ \frac{2}{3}\frac{\gamma-1}{\gamma+1}R(a^{1/\gamma}\hat{g})\sum_{n=0}^{\infty}\frac{3}{3+n}\binom{\gamma^{-1}}{n}\left[\frac{2}{3}\gamma\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\left(\frac{\hat{g}}{a}\right)\right]^n\left[1-\left(\frac{g}{\hat{g}}\right)^{n+3}\right], \quad |\hat{g}/a^*|<1. \end{aligned} \tag{54}$$

The hyperelliptic integral in Eq. (49) can be evaluated by analytical methods. Thus, an analytical discussion of the physical pecularities of the compressible, divergent plasma flow interacting with an inhomogeneous magnetic field is feasible. This will be demonstrated in the following for symmetrical outflows (diffuser):

$$g(-\theta) = g(+\theta) \geqq 0, \quad Q>0, \quad R>0. \tag{55}$$

Equation (55) implies that $g_{\min} = g(\pm\theta_0) = 0$, and $g_{\max} = g(0)$ where $g(0) = 1$ for $\theta = 0$.

IV. Isothermal Plasma Flow

The hyperelliptic integral in Eq. (49) reduces for the isothermal flow to ($\gamma = 1$):

$$\int_{g(0)}^{g(\theta)} \frac{\mathrm{d}g}{[C - 2\varkappa R g - (1-H^2) g^2]^{1/2}} = \pm \int_0^\theta \mathrm{d}\theta \quad (56)$$

in view of the Eqs. (49) and (50). Integration yields

$$[(1-H^2) g + \varkappa R][(\varkappa R)^2 + (1-H^2) C]^{-1/2} = \sin[\pm (1-H^2)^{1/2} \theta + \Phi] \quad (57)$$

where

$$\Phi \equiv \arcsin\{[1-H^2) g(0) + \varkappa R][(\varkappa R)^2 + (1-H^2) C]^{-1/2}\} \quad (58)$$

and

$$C = 2\varkappa R \hat{g} + (1-H^2)\hat{g}^2, [(\varkappa R)^2 + (1-H^2) C]^{1/2} = (1-H^2)\hat{g} + \varkappa R \quad (59)$$

by Equation (51). Application of the boundary conditions in Eq. (44) to Eq. (57) and evaluation of Eq. (57) for $\theta = \hat{\theta}\,(g = \hat{g})$ results in the relations:

$$\varkappa R = \hat{g}[(1-H^2) \cdot \cos(1-H^2)^{1/2} \theta_0] / [1 - \cos(1-H^2)^{1/2} \theta_0], \quad \hat{g} = g(0). \quad (60)$$

Elimination of the constants Φ, C, $\hat{g}$ and $\varkappa R$ by means of Eqs. (58) – (60) from Eq. (57) gives the solution:

$$g(\theta) = \left[1 - \frac{1-\cos(1-H^2)^{1/2}\theta}{1-\cos(1-H^2)^{1/2}\theta_0}\right] g(0). \quad (61)$$

The associated (nondimensional) isothermal flow fields are by Eqs. (34) – (36):

$$u(r,\theta) = \frac{\cos(1-H^2)^{1/2}\theta - \cos(1-H^2)^{1/2}\theta_0}{1-\cos(1-H^2)^{1/2}\theta_0} g(0), \quad (62)$$

$$\varrho(r,\theta) = \frac{1}{r} \frac{(M^2/3R)}{[1-\cos(1-H^2)^{1/2}\theta_0]} \cdot \{3(1-H^2)\cos(1-H^2)^{1/2}\theta_0 + (7+\varepsilon)[\cos(1-H^2)^{1/2}\theta - \cos(1-H^2)^{1/2}\theta_0]\} g(0), \quad (63)$$

$$p(r,\theta) = \frac{1}{r} \frac{(M^2/3R)}{[1-\cos(1-H^2)^{1/2}\theta_0]} \cdot \{3(1-H^2)\cos(1-H^2)^{1/2}\theta_0 + (7+\varepsilon)[\cos(1-H^2)^{1/2}\theta - \cos(1-H^2)^{1/2}\theta_0]\} g(0), \quad (64)$$

where

$$M^2 = \varrho_0 u_0^2 / p_0, \quad \gamma = 1. \quad (65)$$

According to Eqs. (45), (62), and (63), the flow rate Q is for $\gamma = 1$:

$$Q = \frac{M^2}{3R}\left[\frac{g(0)}{1-\cos(1-H^2)^{1/2}\theta_0}\right]^2 \Big\{ (7+\varepsilon)\theta_0 + 2[(7+\varepsilon) - 3(1-H^2)]\theta_0 \cos(1-H^2)^{1/2}\theta_0 - 3[(7+\varepsilon) - 2(1-H^2)] \frac{\sin(1-H^2)^{1/2}\theta_0}{(1-H^2)^{1/2}} \cdot \cos(1-H^2)^{1/2}\theta_0 \Big\}. \quad (66)$$

The Eqs. (62) – (66) are valid for $H \gtrless 1$, and may be rewritten in terms of hyperbolic functions

$$[\cos(1-H^2)^{1/2}\theta = \mathrm{ch}(H^2-1)^{1/2}\theta, \quad \sin(1-H^2)^{1/2}\theta/(1-H^2)^{1/2} = \mathrm{sh}(H^2-1)^{1/2}\theta/(H^2-1)^{1/2}].$$

By means of Bernoulli's rule, one finds in the special case, $H = 1$:

$$u(r,\theta) = \left(1 - \frac{\theta^2}{\theta_0^2}\right) g(0), \quad (67)$$

$$\varrho(r,\theta) = \frac{M^2}{3R}\left[6\theta_0^{-2} + (7+\varepsilon)\left(1 - \frac{\theta^2}{\theta_0^2}\right)\right] g(0), \quad (68)$$

$$p(r,\theta) = \frac{M^2}{3R}\left[6\theta_0^{-2} + (7+\varepsilon)\left(1 - \frac{\theta^2}{\theta_0^2}\right)\right] g(0), \quad (69)$$

and

$$Q = \frac{8}{3}\frac{M^2}{R}\left[\frac{1}{\theta_0} + \frac{2(7+\varepsilon)}{15}\theta_0\right] g(0)^2. \quad (70)$$

The Eqs. (62), (67) indicate that the velocity profile of the isothermal compressible flow varies with increasing H, from a trigonometric ($0 \leqq H < 1$), to a parabolic ($H = 1$), and finally to a hyperbolic ($1 < H < \infty$) structure. It is seen that the velocity profiles become flatter with increasing Hartmann number. Only a symmetrical solution exists, which is unique.

In Eqs. (56) – (64) and Eqs. (66) – (70), $g(0)$ is given in terms of the reference value $g(\overline{\theta}) = 1$ [Eq. (45)] via the solution for $g(\theta)$ in Equation (61). In particular, if the special normalization ($\theta = 0$) in Eq. (25) is chosen, then one has simply $g(0) = 1$.

V. Polytropic Plasma Flow, $1 < \gamma < \infty$

In the general case, $1 < \gamma < \infty$, the hyperelliptic integral in Eq. (49) can be evaluated analytically for large Reynolds numbers, $R \gg 1$, which are exclusively of physical interest. A binomial expansion of

Eq. (33) gives:

$$\begin{aligned}\frac{f}{g} &= \left[a+\frac{2}{3}\,\gamma\,\frac{M^2}{R}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)g\right]^{1/\gamma} \\ &= \left(\frac{\gamma+1}{2}\,\varkappa M^2\right)^{1/\gamma}\left\{1+\sum_{n=1}^{\infty}\binom{\gamma^{-1}}{n}\right. \\ &\qquad \left.\cdot\left[\frac{4}{3}\,\frac{\gamma}{\gamma+1}\left(3+\frac{1+\varepsilon}{1+\gamma}\right)\frac{g}{\varkappa}\right]^n R^{-n}\right\} \qquad (71) \\ &\cong \left(\frac{\gamma+1}{2}\,\varkappa M^2\right)^{1/\gamma} = 1, \quad R \gg 1,\end{aligned}$$

by Equation (42). It is seen that $f(\theta)$ and $g(\theta)$ are equal, $f(\theta)=g(\theta)$ for $R\gg 1$, since $f(\bar{\theta})=g(\bar{\theta})=1$ by Equation (45). Accordingly,

$$M^2=[2/(\gamma+1)]\varkappa^{-1}, \quad 0<M^2\ll R, \quad \varkappa>0. \qquad (72)$$

Equation (72) indicates that the Mach number M has eigenvalue character, since the separation constant $\varkappa$ is an eigenvalue determined by the boundary condition in Equation (44).

Accordingly, only solutions with positive eigenvalues, $\varkappa>0$, have a physical meaning ($M=$ imaginary for $\varkappa<0$). In the same approximation, Eqs. (34) – (36) give for the (nondimensional) plasma fields:

$$u(r,\theta)=r^{-(\gamma-1)/(\gamma+1)}\,g(\theta), \qquad (73)$$

$$\varrho(r,\theta)=r^{(\gamma-1)/(\gamma+1)-1}, \qquad (74)$$

$$p(r,\theta)=r^{\gamma((\gamma-1)/(\gamma+1)-1)}; \quad R\gg 1. \qquad (75)$$

In the limit of large Reynolds numbers, the density and pressure fields depend only on the radial (r) but no longer on the transverse (θ) coordinates, in agreement with the so-called boundary-layer approximation [14].

In view of Eq. (71), Eq. (49) reduces to the elliptic integral [cf. Eq. (43)]:

$$\int_{g(\tilde{\theta})}^{g(\theta)}\frac{dg}{\left\{\bar{C}-g^3-\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right]g^2-2\,\frac{\varkappa R}{\Omega}\,g\right\}^{1/2}} = \pm\sqrt{\Omega}\int_{\tilde{\theta}}^{\theta}d\theta \qquad (76)$$

where

$$\Omega\equiv\frac{2}{3}\left(\frac{\gamma-1}{\gamma+1}\right)R>0, \qquad (77)$$

$$\bar{C}\equiv C/\Omega\geqq 0. \qquad (78)$$

The integration constant is positive,

$$\bar{C}=(1/\Omega)\,(dg/d\theta)^2_{\theta=\pm\theta_0}\geqq 0,$$

since $g(\theta=\pm\theta_0)=0$ by Equation (44). Equation (76) is rewritten in the form

$$\pm\sqrt{\Omega}\int_{\tilde{\theta}}^{\theta}d\theta=\int_{g(\tilde{\theta})}^{g(\theta)}\frac{dg}{[(-1)\,(g-g_1)\,(g-g_2)\,(g-g_3)]^{1/2}} \qquad (79)$$

where

$$g_1+g_2+g_3=-\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right], \qquad (80)$$

$$g_1g_2+g_1g_3+g_2g_3=2\,\frac{\varkappa R}{\Omega}>0, \qquad (81)$$

$$g_1g_2g_3=\bar{C}\geqq 0. \qquad (82)$$

If $\hat{g}$ designates one of the roots (g_1, g_2, g_3) of the cubic in Eq. (76), then

$$\bar{C}=\hat{g}^3+\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right]\hat{g}^2+2\,\frac{\varkappa R}{\Omega}\,\hat{g}. \qquad (83)$$

Accordingly, the roots of the trinomial are explicitly in terms of $\hat{g}$

$$g=\hat{g}, \quad g=g_+, \quad g=g_-;$$
$$g_+\geqq g_- \quad \text{if} \quad g_\pm=\text{real}, \qquad (84)$$

where

$$\begin{aligned} g_\pm = \frac{1}{2}\Bigg[&-\left\{\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right]+\hat{g}\right\} \\ &\pm\left(\left\{\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right]+\hat{g}\right\}^2\right. \qquad (85) \\ &\left.-8\,\frac{\varkappa R}{\Omega}-4\hat{g}\left\{\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right]+\hat{g}\right\}\right)^{1/2}\Bigg].\end{aligned}$$

Equation (76) or Eq. (80) leads to different types of outflow solutions as will be explained by a simple physical argument. In differential form, Eq. (76) can be written as

$$\tfrac{1}{2}\,(dg/d\theta)^2+V(g)=0 \qquad (86)$$

where

$$\begin{aligned}V(g)=-\frac{1}{2}\,\Omega\,\Bigg\{&\bar{C}-g^3 \\ &-\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right]g^2-\frac{2\varkappa Rg}{\Omega}\Bigg\}\leqq 0 \qquad (87)\end{aligned}$$

is negative, since $(dg/d\theta)^2\geqq 0$ for any $-\theta_0\leqq\theta\leqq+\theta_0$. Equation (86) may be interpreted as the energy conservation equation for a fictitious particle of mass $m=1$, kinetic energy $^1/_2(dg/d\theta)^2$, potential energy $V(g)$, and total energy $E=0$ ($g\triangleq$ displacement, $\theta\triangleq$ time). The locations of the minimum (+) and maximum (−) of the potenial $V(g)$ are given by $\mathrm{d}V(g)/\mathrm{d}g = 0$ as

$$g^{(\pm)} = \frac{1}{2}\left[-\frac{2}{3\,\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]\right.$$
$$\left.\pm\left(\left|\frac{2}{3\,\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]\right|^2 - \frac{8}{3}\frac{\varkappa R}{\Omega}\right)^{1/2}\right]$$
$$g^{(+)} > g^{(-)}, \quad \text{if} \quad g^{(\pm)} = \text{real}, \tag{88}$$

since

$$\frac{2}{3\,\Omega}\left[\frac{\mathrm{d}^2V(g)}{\mathrm{d}g^2}\right]_{g=g^{(\pm)}} = \pm\left(\left|\frac{2}{3\,\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]\right|^2 - \frac{8}{3}\frac{\varkappa R}{\Omega}\right)^{1/2}. \tag{89}$$

Accordingly, the maximum of $V(g)$ precedes the minimum of $V(g)$ as shown in Figure 2. In the possible cases $(1-2)$, the fictitious particle starts at the origin $g=0$ for $\theta = -\theta_0$ and returns to origin $g=0$ for $\theta = +\theta_0$, where the allowed path $g(\theta)$ is determined by the condition $V(g) \leqq 0$ [Equation (84)]. The case 3) requires negative $\varkappa$-values, i. e. is physically not realizable (M = imaginary).

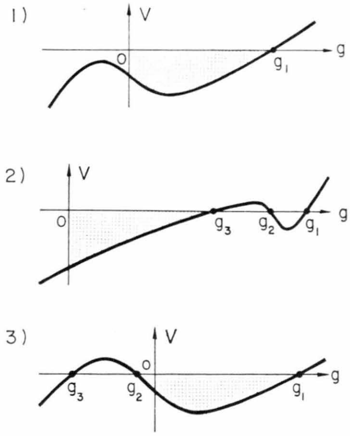

Fig. 2. Fictitious potential V versus "particle position" g for the Cases 1, 2, 3 (qualitative).

1. Case: $g_1 > 0$, $g_{2,3}$ = Conjugate Complex

If g_2 is the complex conjugate of g_3, then $g_1 > 0$ since $\bar{C} = g_1 g_2 g_3 \geqq 0$. Evidently, it is $g_1 \equiv \hat{g}$ and $g_{2,3} \equiv g_\pm$ [Eqs. (84) – (85)]. The associated potential curve $V(g)$ is shown in Figure 2.1. It is seen that $V(g) \leqq 0$ is satisfied for $-\infty < g(\theta) \leqq g_1$. Hence, pure outflows are restricted to the interval

$$0 \leqq g(\theta) \leqq g_1, \quad g_1 = \hat{g} = g(0). \tag{90}$$

Equation (79) is reduced by means of the transformation $[\tilde{\theta} \equiv \hat{\theta} = 0,\ g(\tilde{\theta}) = g(0)]$

$$g(\theta) = g(0) - \lambda^2[(1-\cos\varphi)/(1+\cos\varphi)],$$
$$0 \leqq \varphi < \pi, \tag{91}$$

to

$$\pm\sqrt{\Omega}\,\theta = -\lambda^{-1}F(\varphi, k), \tag{92}$$

where

$$\lambda = \left\{3\,g(0)^2 + \frac{2}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]g(0) + 2\,\frac{\varkappa R}{\Omega}\right\}^{1/4}, \tag{93}$$

$$k^2 = \tfrac{1}{2} + \tfrac{1}{4}\lambda^{-2}\left\{3\,g(0) + \frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2 - H^2\right]\right\}. \tag{94}$$

The amplitude $g(\theta)$ of the velocity field obtains by inversion of Eq. (92) and substitution of $\varphi(\theta)$ into Eq. (91) as:

$$g(\theta) = g(0) - \lambda^2\frac{[1-\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta;k)]}{[1+\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta;k)]}. \tag{95}$$

The separation constant $(\varkappa)$ in this solution is determined (implicitly) by the boundary condition in Eq. (44), i. e., the transcendental equation

$$\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta_0;k) = [\lambda^2 - g(0)]/[\lambda^2 + g(0)]. \tag{96}$$

The flow rate is in Case 1:

$$Q = 2\,\theta_0[g(0) - \lambda^2]$$
$$-4\frac{\lambda}{\sqrt{\Omega}}\left[\frac{\mathrm{sn}(\sqrt{\Omega}\,\lambda\,\theta_0;k)\,\mathrm{dn}(\sqrt{\Omega}\,\lambda\theta_0;k)}{1+\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta_0;k)} - E(\varphi_0;k)\right], \tag{97}$$

where [15] $\varphi_0 = \mathrm{am}(\sqrt{\Omega}\,\lambda\,\theta_0;k)$. In Eqs. (90) – (97), $g(0)$ is determined through the normalization, i. e., $g(0) = 1$ for $\bar{\theta} = 0$.

The flow type 1) exists, since $2\varkappa R/\Omega = |g_{2,3}|^2 + 2\,\mathrm{Re}\,g_{2,3} > 0$ $(\bar{\theta} = 0)$ by Eq. (80), i. e., Eq. (72) is satisfied. Since $\mathrm{Re}\,g_{2,3} < g_1$ and $g_1 = g(0) = 1\ (\bar{\theta} = 0)$, Case 1 requires Hartmann numbers H below a critical value [see Eq. (80), $\bar{\theta} = 0$]

$$0 \leqq H^2 < [2\,\gamma/(\gamma+1)]^2$$
$$+2[(\gamma-1)/(\gamma+1)]R(0). \tag{98}$$

If the roots g_1, g_2, and g_3 are all real, the flow type 2 exists. The numbering of the roots is chosen such that

$$g_1 \geqq g_2 \geqq g_3\,. \tag{99}$$

2. Case: $g_3 = g(0) > 0$, $g_{1,2} > 0$.

If $g_3 = g(0) > 0$, than $g_{1,2} > 0$ is possible since $\bar{C} = g_1 g_2 g_3 \geqq 0$. The associated potential curve $V(g)$ is shown in Figure 2.2. It is $V(g) \leqq 0$ for $-\infty < g(\theta) \leqq g_3$. Accordingly, pure outflows are limited to the interval

$$0 \leqq g(\theta) \leqq g_3, \quad g_3 = g(0). \tag{100}$$

Equation (79) is reduced by means of the transformation $[\tilde{\theta}=\theta_3=0,\, g(\tilde{\theta})=g(0)]$

$$g(\theta)=g_2-[g_2-g(0)]\cos^{-2}\varphi, \quad 0\leqq\varphi\leqq\pi/2 \tag{101}$$

to

$$\pm\sqrt{\Omega}\,\theta=-\lambda^{-1}F(\varphi,k), \tag{102}$$

where

$$\lambda=\tfrac{1}{2}[g_1-g(0)]^{1/2}, \tag{103}$$

$$k^2=(g_1-g_2)/[g_1-g(0)]. \tag{104}$$

The amplitude $g(\theta)$ of the velocity field obtains by inversion of Eq. (102) and substitution of $\varphi(\theta)$ into Eq. (101) as:

$$g(\theta)=g_2-[g_2-g(0)]\,\mathrm{cn}^{-2}(\sqrt{\Omega}\,\lambda\,\theta;k). \tag{105}$$

The eigenvalue $\varkappa$ is determined by the boundary condition in Eq. (44),

$$\mathrm{cn}^2(\sqrt{\Omega}\,\lambda\,\theta_0,k)=[g_2-g(0)]/g_2\geqq 0. \tag{106}$$

The flow rate is in Case 2:

$$Q=2\,\theta_0\,g(0)-2\,\frac{g_1-g(0)}{\sqrt{\Omega}\,\lambda}\cdot\left[\frac{\mathrm{sn}(\sqrt{\Omega}\,\lambda\,\theta_0;k)\,\mathrm{dn}(\sqrt{\Omega}\,\lambda\,\theta_0;k)}{\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta_0;k)}-E(\varphi_0;k)\right] \tag{107}$$

where[15] $\varphi_0=\mathrm{am}(\sqrt{\Omega}\,\lambda\,\theta_0,k)$. In Eqs. (100) – (107), g_1 and g_2 are functions of $\varkappa$ defined by Eqs. (80) – (81),

$$g_1+g_2+g(0)=-\frac{1}{\Omega}\left[4\left(\frac{\gamma}{\gamma+1}\right)^2-H^2\right], \tag{108}$$

$$g_1\,g_2+(g_1+g_2)g(0)=2\,\varkappa\,R/\Omega>0, \tag{109}$$

where $g(0)$ is given by the normalization, i.e. $g(0)=1$ for $\overline{\theta}=0$.

The flow type 2 satisfies Eq. (72), since $\varkappa>0$ for $g_{1,2,3}>0$ by Eq. (109). Since $g_{1,2}<g_3$, and $g_3=g(0)=1\,(\overline{\theta}=0)$, Case 2 requires Hartmann numbers H above a critical value [see Eq. (108), $\overline{\theta}=0$],

$$\left(\frac{2\gamma}{\gamma+1}\right)^2+2\,\frac{\gamma-1}{\gamma+1}\,R(0)<H^2<\infty. \tag{110}$$

A comparison of Eq. (98) and (110) indicates that the flow types 1) and 2) appear in adjacent intervals of the Hartmann number H, $0\leqq H<H_c$, and $H_c<H<\infty$, where

$$H_c^2=(2\gamma/\gamma+1)^2+2[(\gamma-1)/(\gamma+1)]R(0).$$

Thus, the similarity transformation leads to compressible flow solutions for all possible Hartmann numbers, $0\leqq H<\infty$ $(0<R<\infty,\ 0<\gamma<\infty)$. The limiting solutions with vanishing small eigenvalues, $\varkappa\cong 0$, have to be obtained numerically from Eq. (49), since the expansion for large R breaks down for $\varkappa\to 0$.

In Fig. 2, the remaining case 3 has only a formal mathematical meaning. Since $g_1=g(0)=1\,(\overline{\theta}=0)$ and $g_{2,3}<0$ in case 3, the Eqs. (80) and (81) can be satisfied (simultaneously) only for $\varkappa<0$, but not for $\varkappa>0$. Accordingly, the case 3 would imply imaginary Mach numbers M [Eq. (72)], and is, therefore, of no physical meaning. It is interesting that for the incompressible flow solutions with negative $\varkappa$-values are considered to be of physical relevance [2,3], since for $\gamma=\infty$ a similarity transformation exists with a positive overpressure so that $p(r,\theta)>0$ for $\varkappa<0\,(|\theta|\leqq\theta_0,\ r\geqq r_1)$.

VI. Discussion

It is experimentally known that laminar, compressible flows cannot be realized in diverging ducts (diffusers, wind tunnels, nozzles) at arbitrary Mach numbers M for a fixed duct angle θ_0. With increasing Reynolds number R, laminar outflows are observed only for sufficiently small duct angles θ_0 because of boundary-layer separation. In the case of plasmas, laminar flows have been observed for large dunct angles, $0\leqq\theta_0<\pi$, if the magnetic field B_θ is larger than a critical value. These observations can be explained quantitatively by further inspection of the solutions derived under Case 1 and Case 2.

1. Case: An infinite number of real eigenvalues $\varkappa>0$ exists by Eq. (96) in Case 1, since

$$-1\leqq\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta_0,k)\leqq+1$$

is a periodic function of $\varkappa$ [with period $4K(k)$ which varies with $\varkappa$, $k=k(\varkappa)$] and

$$-1<(\lambda^2-1)/(\lambda^2+1)\leqq+1 \quad \text{for } 0<\varkappa\leqq\infty.$$

For extremely large eigenvalues, $\varkappa^{1/4}\gg 1$, one has asymptotically

$$\lambda=(2R/\Omega)^{1/4}\varkappa^{1/4}\gg 1, \quad \varkappa^{1/4}\gg 1, \tag{111}$$

$$k=2^{-1/2}, \quad \varkappa^{1/4}\gg 1, \tag{112}$$

by Eqs. (93) and (94). Equation (95) gives as solution at large eigenvalues:

$$g(\theta)=g(0)-\lambda^2\,\frac{[1-\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta,2^{-1/2})]}{[1+\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta,2^{-1/2})]}, \quad \varkappa^{1/4}\gg 1 \tag{113}$$

where

$$\lambda = [4\,K(2^{-1/2})/\sqrt{\Omega}\,\theta_0]\,s \gg 1\,,$$
$$s = 1, 2, 3, \ldots, \quad \infty, \quad \varkappa^{1/4} \gg 1\,, \tag{114}$$

by Equation (96). It is seen that the solutions with large eigenvalues, $\varkappa^{1/4} \gg 1$, are not only negative ($g<0$) for sufficiently large θ but also diverge (e.g., $|g| = \infty$ at $\theta = \theta_0/2$). Obviously, they have no physical meaning.

The cosine amplitude function in Eq. (96) has zeroes at $\sqrt{\Omega}\,\lambda\,\theta_0 = s'\,K(k)$, $s' = 1, 3, 5, \ldots, \infty$. Accordingly, only the first, positive root, $\varkappa_1 > 0$, of Eq. (96) gives a positive solution, $g(\theta) \geqq 0$, provided that the duct angle is below a critical value,

$$\theta_0 < \theta_0{}^{\mathrm{c}}\,, \quad \theta_0{}^{\mathrm{c}} = \theta_0(\varkappa = 0)\,, \tag{115}$$

where $\theta_0{}^{\mathrm{c}}$ has to be determined numerically from Eqs. (43) – (45) for $a = 0$. The numerical value of $\varkappa_1$ has to be determined by iteration from Eq. (96) for the given flow parameters R, $H < H_{\mathrm{c}}$, and $\theta_0 < \theta_0{}^{\mathrm{c}}$. Equation (95) gives the associated flow solution, which is positive and unique.

According to Eqs. (43) – (95), the flow exhibits a fundamental invariance property. If $R \gg 1$, $0 \leqq H < H_{\mathrm{c}}$, $\gamma > 1$, and $0 < \theta_0 < \theta_0{}^{\mathrm{c}}$ are varied, but such that the combinations

$$I_1 \equiv \left(\frac{\gamma-1}{\gamma+1}\,R\right)^{1/2} \theta_0\,, \quad I_2 \equiv \frac{\gamma+1}{\gamma-1}\left[H^2 - \left(\frac{2\,\gamma}{\gamma+1}\right)^2\right] \Big/ R \tag{116}$$

remain unchanged, then $g(\theta/\theta_0)$, and

$$x \equiv [(\gamma+1)/(\gamma-1)]\,\varkappa_1\,, \quad \mathsf{M} \equiv (\gamma-1)^{1/2} M \tag{117}$$

are invariant. This means that the solution $g(\theta)$ depends only on the normalized angle θ/θ_0 and the parameters I_1 and I_2.

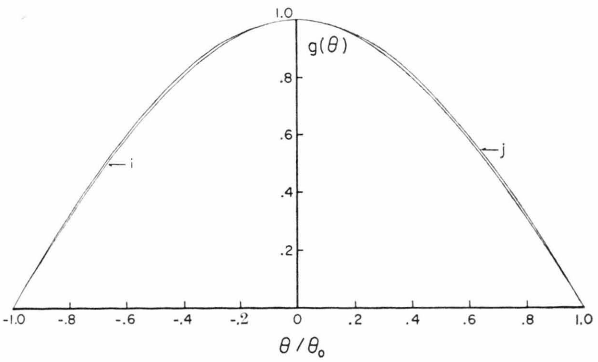

Fig. 3. $g(\theta/\theta_0)$ für $I_1 = 25 \cdot 10^{1/2}(\pi/180)$ and i) $I_2 = 0$ and j) $I_2 = 4 \cdot 10^{-1}$

In Fig. 3, $g(\theta/\theta_0)$ is shown for the cases $I_1 = 25\sqrt{10}(\pi/180)$ and $I_2 = 0$, $4 \cdot 10^{-1}$. In Fig. 4, $g(\theta/\theta_0)$ is shown for the cases $I_1 = 50(\pi/180)$ and $I_2 = 0$, $4 \cdot 10^{-1}$. It is, e.g., $I_1 = 25\sqrt{10}(\pi/180)$ for

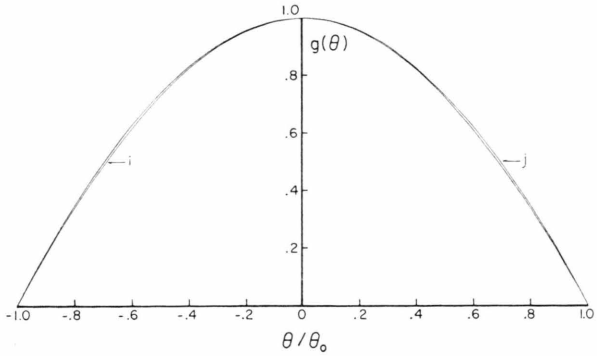

Fig. 4. $g(\theta/\theta_0)$ for $I_1 = 50\ (\pi/180)$ and i) $I_2 = 0$ and j) $I_2 = 4 \cdot 10^{-1}$.

$R = 10^3$, $\theta_0 = \pi/36$, $\gamma = 5/3$, and $I_1 = 50(\pi/180)$ for $R = 10^4$, $\theta_0 = \pi/180$, $\gamma = 5/3$. Further, if $\gamma = 5/3$, it is $I_2 = 0$ for $H^2/R = 0 + (5/4)^2/R \cong 0$, and $I_2 = 4 \cdot 10^{-1}$ for $H^2/R = 10^{-1} + [(5/4)^2/R] \cong 10^{-1}$. The numerical values of the eigenvalues $(\varkappa_1, x)$ and the associated Mach numbers (M, M) are given for the above values of I_1 and I_2 in Table 1.

Table I. Eigenvalues $(\varkappa_1, x)$ and Mach numbers (M, M) for given parameters I_1 and I_2.

I_1	I_2	x	M	$\varkappa_1(\gamma=5/3)$	$M(\gamma=5/3)$
$25\sqrt{10}(\pi/180)$	0	0.3520	2.384	0.0880	2.920
$25\sqrt{10}(\pi/180)$	$4 \cdot 10^{-1}$	0.6680	1.730	0.1670	2.119
$50(\pi/180)$	0	1.9064	1.024	0.4766	1.254
$50(\pi/180)$	$4 \cdot 10^{-1}$	2.2332	0.946	0.5583	1.159

The Figs. 3 – 4 indicate that in Case 1 ($H < H_{\mathrm{c}}$) the velocity profiles become flatter with increasing H. This effect of the magnetic field is, however, small as long as H is noticeably below H_{c}. If R is increased, then solutions exist only for smaller values of θ_0, since $\theta_0 < \theta_0{}^{\mathrm{c}}$ by Equation (115).

2. Case: Equation (106) indicates that in general more than one real eigenvalue, $\varkappa > 0$, exists in Case 2. Since $\mathrm{cn}(\sqrt{\Omega}\,\lambda\,\theta_0, k) = 0$ for $\sqrt{\Omega}\,\lambda\,\theta_0 = s'\,K(k)$, $s' = 1, 3, 5, \ldots, \infty$, a positive solution, $g(\theta) \geqq 0$, exists only for the first root $\varkappa_1 > 0$ of Equation (106). The latter has no roots for $g_2 < g(0)$. Since

$$(g_2 - g(0))/g_2 = 0\,, \quad \text{and} \quad k^2 = 1 \ \text{for} \ g_2 = g(0)$$

and

$$\mathrm{cn}^2(\sqrt{\Omega}\,\lambda\,\theta_0,1)=\cosh^{-2}(\sqrt{\Omega}\,\lambda\,\theta_0)\cong 0 \quad \text{if } \sqrt{\Omega}\,\lambda\,\theta_0 \gg 1,$$

the first root $\varkappa_1$ is given (in excellent approximation) by the relation $g_2(\varkappa_1)=g(0)$, i. e.

$$\varkappa_1=\frac{\Omega}{R}\left\{\frac{1}{\Omega}\left[H^2-\left(\frac{2\gamma}{\gamma+1}\right)^2\right]-\frac{3}{2}g(0)\right\}g(0)>0, \quad \sqrt{\Omega}\,\lambda_1\,\theta_0\gg 1, \qquad (118)$$

by Eqs. (108) and (109). In the same approximation, the flow solution is by Eqs. (105) and (106):

$$g(\theta)=g(0)\left\{1-\left[\frac{\cosh(\sqrt{\Omega}\,\lambda_1\,\theta)}{\cosh(\sqrt{\Omega}\,\lambda_1\,\theta_0)}\right]^2\right\},\ \sqrt{\Omega}\,\lambda_1\,\theta_0\gg 1 \qquad (119)$$

where

$$\lambda_1=\frac{1}{2}\left\{\frac{1}{\Omega}\left[H^2-\left(\frac{2\gamma}{\gamma+1}\right)^2\right]-3g(0)\right\}^{1/2}>0 \qquad (120)$$

by Eqs. (103) and (108). The approximations in Eqs. (118) – (119) are based on the restriction $\sqrt{\Omega}\,\lambda_1\,\theta_0 \gg 1$ which is satisfied in many cases if λ_1 and θ_0 are not too small, since $\Omega \sim R \gg 1$. If $\sqrt{\Omega}\,\lambda_1\,\theta_0 \lesssim 1$, $\varkappa_1$ has to be computed by iteration from Equation (106). The associated eigensolution is defined by the original Equation (105).

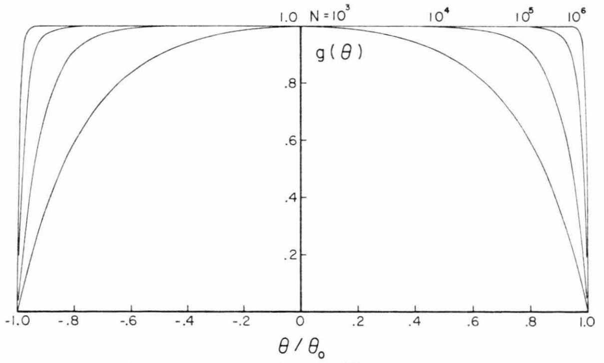

Fig. 5. $g(\theta/\theta_0)$ for $I_1=5(\pi/180)N^{1/2}$, $N=10^3-10^6$, and $I_2=4$.

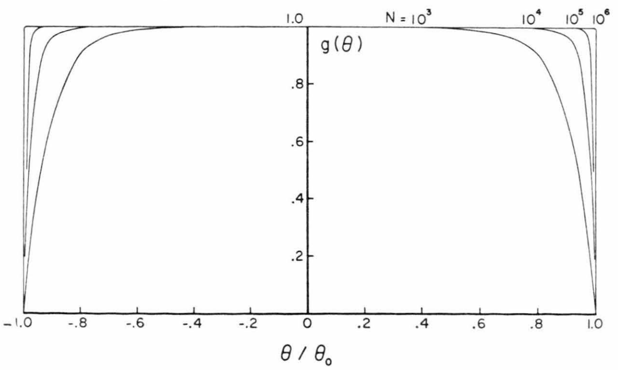

Fig. 6. $g(\theta/\theta_0)$ for $I_1=5(\pi/180)N^{1/2}$, $N=10^3-10^6$, and $I_2=20$.

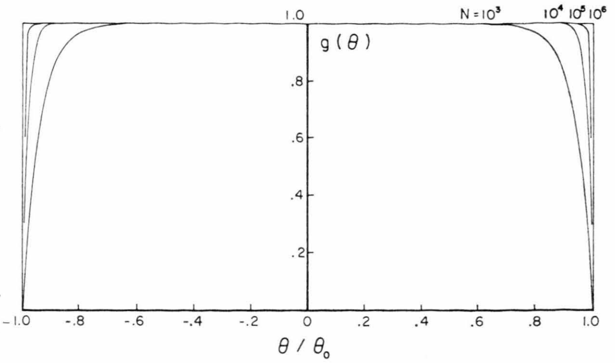

Fig. 7. $g(\theta/\theta_0)$ for $I_1=5(\pi/180)N^{1/2}$, $N=10^3-10^6$, and $I_2=40$.

The invariance property of the flow is obvious from the Eqs. (103) – (105): $g(\theta/\theta_0)$, x, and M are invariant with respect to variations of $R\gg 1$, $H>H_c$, $\gamma>1$, and $0<\theta_0\leqq\pi$, which leave I_1 and I_2 [Eq. (116)] unchanged. Equation (118) indicates that x and M [Eq. (117)] depend only on I_2 in the limit $\sqrt{\Omega}\,\lambda_1\,\theta_0\gg 1$.

In Figs. 5, 6, and 7, $g(\theta/\theta_0)$ is shown for $I_1=5(\pi/180)N^{1/2}$, $N=10^3-10^6$, and $I_2=4$, 20, 40. In Figs. 8, 9, and 10, $g(\theta/\theta_0)$ is shown for $I_1=(45/2)(\pi/180)N^{1/2}$, $N=10^3-10^6$, and $I_2=4$, 20, 40. It is, e. g., $I_1=5(\pi/180)N^{1/2}$ for $R=N$, $\theta_0=\pi/18$, $\gamma=5/3$; and $I_1=(45/2)(\pi/180)N^{1/2}$ for $R=N$, $\theta_0=\pi/4$, $\gamma=5/3$. Further, $I_2=4$, 20, 40 implies that $H^2/R\cong 1$, 5, 10 if $\gamma=5/3$, since $(2\gamma/\gamma+1)^2/R\ll 1$. The numerical values of $\varkappa_1$, x and M, M are given for $I_1=5(\pi/180)N^{1/2}$, $N=10^3-10^6$, and $I_2=5$, 20, 40 in Table 2. [They are practically the same for $I_1=(45/2)(\pi/180)N^{1/2}$ except in the case $N=10^3$; see Equation (118).]

The Figs. 5 – 7 and the Figs. 8 – 10 indicate that in Case 2 ($H>H_c$) the flatness of the velocity profile increases and the thickness of the boundary layer shrinks considerably with increasing I_2 or H^2/R. This effect is more pronounced at larger values of I_1. The flow solution is unique and exists for all values of θ_0, $0<\theta_0<\pi$.

Table 2. Eigenvalues ($\varkappa_1$, x) and Mach numbers (M, M) for given parameters I_1 and I_2.

$I_1/5$ $(\pi/180)$	I_2	x	M	$\varkappa_1$ $(\gamma=5/3)$	M $(\gamma=5/3)$
$10^{3/2}$	5	3.0692	0.808	0.7673	0.989
$10^{3/2}$	20	18.9936	0.324	4.7484	0.397
$10^{3/2}$	40	38.9936	0.226	9.7484	0.277
10^2	5	3.0000	0.816	0.7500	1.000
10^2	20	19.0000	0.324	4.7500	0.397
10^2	40	39.0000	0.226	9.7500	0.277

(No changes for $N>10^2$)

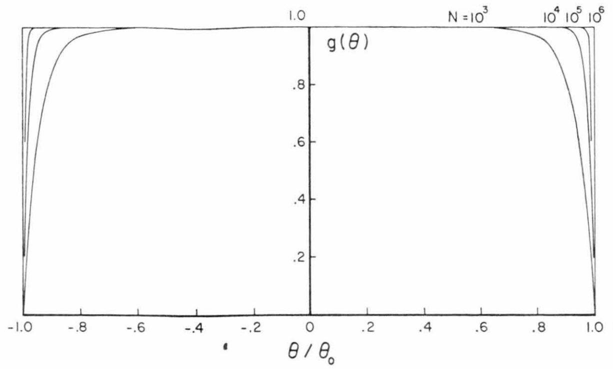

Fig. 8. $g(\theta/\theta_0)$ for $I_1 = (45/2)\,(\pi/180)\,N^{1/2}$, $N = 10^3 - 10^6$, and $I_2 = 4$.

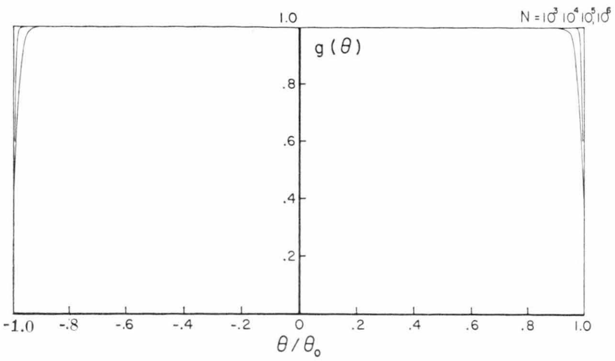

Fig. 10. $g(\theta/\theta_0)$ for $I_1 = (45/2)\,(\pi/180)\,N^{1/2}$, $N = 10^3 - 10^6$, and $I_2 = 40$.

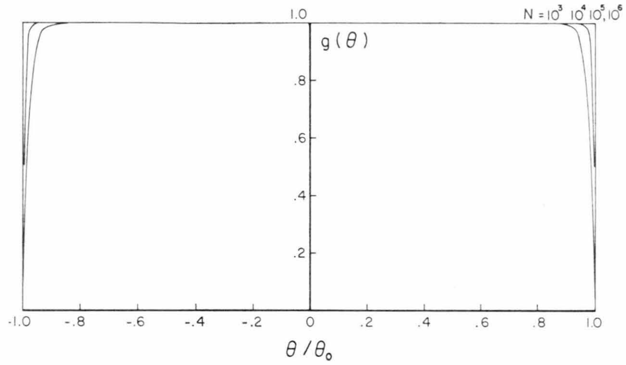

Fig. 9. $g(\theta/\theta_0)$ for $I_1 = (45/2)\,(\pi/180)\,N^{1/2}$, $N = 10^3 - 10^6$, and $I_2 = 20$.

VII. Conclusions

A closed form similarity solution is feasible for the compressible Jeffery-Hamel outflow (diffuser) of a plasma across an azimuthal magnetic field. The solution is different in type depending on whether the Hartmann number is 1) $H < H_c$ or 2) $H > H_c$. These solutions exist for a limited range of duct angles, $0 < \theta_0 < \theta_0^c < \pi$, in Case 1) and for all duct angles, $0 < \theta_0 < \pi$, in Case 2 (stabilizing effect of the magnetic field). The velocity profile and boundary layer are affected weakly in Case 1), and strongly in Case 2) by the magnetic field. The solutions depend only on the normalized coordinates $\tilde{r} = r/r_0$, $\theta = \theta/\theta_0$, and the parameters $I_1 = I_1$ (R, γ, θ_0) and $I_2 = I_2$ (R, H, γ), i.e. are the same if R, H, γ, and θ_0 are varied (within their limits) such that I_1 and I_2 are left unchanged. The Mach number M occurs as an eigenvalue. The (outflow) solutions are unique, since the higher eigenvalues lead to physically meaningless solutions.

The problem arose in connection with experiments on compressible plasma flows across inhomogeneous magnetic fields in diverging ducts [16−17].

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